Deterministic dynamics in the minority game

P. Jefferies, M. L. Hart, and N. F. Johnson
Physics Department, Clarendon Laboratory, Oxford University, Oxford OX1 3PU, United Kingdom
(Received 13 March 2001; published 11 December 2001)

The minority game (MG) behaves as a stochastically disturbed deterministic system due to the coin toss invoked to resolve tied strategies. Averaging over this stochasticity yields a description of the MG's deterministic dynamics via mapping equations for the strategy score and global information. The strategy-score map contains both restoring-force and bias terms, whose magnitudes depend on the game's quenched disorder. Approximate analytical expressions are obtained and the effect of "market impact" is discussed. The global-information map represents a trajectory on a de Bruijn graph. For small quenched disorder, a Eulerian trail represents a stable attractor. It is shown analytically how antipersistence arises. The response to perturbations and different initial conditions is also discussed.

DOI: 10.1103/PhysRevE.65.016105 PACS number(s): 02.50.Le, 05.40.−a, 87.23.Ge, 01.75.+m

I. INTRODUCTION

The minority game (MG) introduced by Challet and Zhang [1] offers possibly the simplest paradigm for a complex, dynamical system comprising many competing agents. Models based on the minority game concept have a broad range of potential applications, for example, financial markets, biological systems, crowding phenomena, and routing problems [2]. There have been many studies of the statistical properties of the MG [1–4,6–15], which treat the game as a quasistochastic system.

In this paper we examine the MG from a different perspective by treating it as a primarily *deterministic* system and then exploring the rich dynamics that result. Our desire to look at microscopic dynamical properties, as opposed to global statistics, is motivated by the fact that the physical systems we are interested in modeling are only realized once (e.g., the time evolution of a financial market). Only limited insight is therefore available from taking configuration averages in such cases. In addition it is of great interest to examine transient effects such as the response of the system to perturbations and the mechanisms that determine the game's trajectories in time. We find that we are able to provide a description of the resulting deterministic dynamics via mapping equations, and can hence investigate these important effects. The outline of the paper is as follows: after briefly discussing the MG in the remainder of this section, Sec. II examines the MG as a functional map. Section III focuses on the effect of the underlying ("quenched") disorder arising from unequal population of the strategy space. Section IV discusses the dynamics of the game on a de Bruijn graph. Section V provides the conclusions.

The most basic formulation of a MG comprises an odd number of agents N who at each turn of the game choose between two options "0" and "1" [1,2]. These options could be used to represent buy/sell, choose road A/road B, etc. The aim of the agents is common: to choose the least subscribed option, the "minority" group. At the end of each turn of the game, the winning decision corresponds to the minority group and is announced to all the agents. The agents have a *memory* of m bits, hence they can recall the last m winning decisions. The *global information* μ available to each and every agent is therefore a binary word m bits long, hence μ belongs to the set $\{0,1,\ldots,P-1\}$ where $P=2^m$. In order to make a decision about which option to choose, each agent is allotted s *strategies* at the outset of the game, which cannot be altered during the game. Each strategy R maps every possible value of μ to a prediction $a_R^\mu \in \{-1,1\}$, where $-1 \Rightarrow (\text{option } 0)$ and $1 \Rightarrow (\text{option } 1)$. There are 2^{2^m} different possible binary strategies. However, many of the strategies in this space are similar to one another, i.e., they are separated by a small Hamming distance. It has been shown [3] that the principle features of the MG are reproduced in a smaller reduced strategy space (RSS) of 2^{m+1} strategies, in which any two strategies are separated by a Hamming distance of 2^m or 2^{m-1}, i.e., the two strategies are *anticorrelated* or *uncorrelated*, respectively.

The agents follow the prediction of their historically best performing strategy. They measure this performance by rewarding strategies with the correct mapping of global information to winning decision and penalizing those with an incorrect mapping. Strategies are scored in this manner irrespective of whether they are played. As each agent will reward and penalize the same strategy in the same way, there is a common set of strategy scores that are collected together to form the *strategy-score vector* $\underline{S}$. The common perception of a strategy's success or failure will lead to agents deciding to use or avoid the same strategy in groups—this leads to crowd behavior as analyzed in Refs. [4,6].

II. MG AS A FUNCTIONAL MAP

The minority game is often introduced heuristically as a set of rules determining the update of the agents' strategies and the global information. It can however easily be cast into a functional map, which reproduces the game when iterated. Moreover, this functional map can be iterated *without* having to keep track of the labels for individual agents. We achieve this by introducing a formalism that groups together agents who hold the same combination of strategies, and hence respond in an identical way to all values of the global-information set $\mu=\{0,1,\ldots,P-1\}$. This grouping is achieved via the tensor $\underline{\underline{\Omega}}$, which is initialized at the outset of the game and quantifies the particular quenched disorder for that game [4]. $\underline{\underline{\Omega}}$ is s dimensional with rows and columns of length $2P$ (in the RSS) such that entry $\underline{\underline{\Omega}}_{R_1,R_2,\ldots}$ is the number of agents holding strategies $\{R_1,R_2,\ldots\}$. The entries of $\underline{\underline{\Omega}}$ (and also of the strategy-score vector $\underline{S}$) are ordered by increasing decimal equivalent. For example, strategies from the RSS for $m=2$ are ordered {0000, 0011, 0101, 0110, . . . }, therefore strategy R is anticorrelated to strategy $2P+1-R$. $\underline{\underline{\Omega}}$ is randomly filled with uniform probability such that

$$\sum_{R,R',\ldots} \underline{\underline{\Omega}}_{R,R',\ldots}=N.$$

It is useful to construct a configuration of this tensor, $\underline{\underline{\Psi}}$, which is symmetric in the sense that $\underline{\underline{\Psi}}_{\{R_1,R_2,\ldots\}}=\underline{\underline{\Psi}}_{p\{R_1,R_2,\ldots\}}$, where $p\{R_1,R_2,\ldots\}$ is any permutation of strategies $R_1,R_2,\ldots$. For $s=2$ we let $\underline{\underline{\Psi}}=\frac{1}{2}(\underline{\underline{\Omega}}+\underline{\underline{\Omega}}^T)$ [16]. Now we proceed to a formula for the attendance A of the MG (i.e., the sum of all the agents' predictions and hence actions),

$$A=\underline{a}^{\mu}\cdot\underline{n}=\sum_{R=1}^{2P} a_R^{\mu} n_R\,, \tag{1}$$

where a_R^{μ} is the response of strategy R to global information μ and n_R is the number of agents playing strategy R. We can define n_R in terms of the strategy-score vector $\underline{S}$ and $\underline{\underline{\Psi}}$ and hence rewrite Eq. (1) to give the following for $s=2$:

$$A(\underline{S},\mu)=\sum_{R=1}^{2P} a_R^{\mu}\sum_{R'=1}^{2P}[1+\mathrm{sgn}(S_R-S_{R'})]\underline{\underline{\Psi}}_{R,R'} + \sum_{R\neq R'}^{2P} a_R^{\mu}\delta_{S_R,S_{R'}}\left(\mathrm{bin}[2\underline{\underline{\Psi}}_{R,R'},\tfrac{1}{2}]-\underline{\underline{\Psi}}_{R,R'}\right), \tag{2}$$

where bin $[n,p]$ is a sample from a binomial distribution of n trials with probability of success p. Here the constraint $\mathrm{bin}[2\underline{\underline{\Psi}}_{R,R'},\frac{1}{2}]+\mathrm{bin}[2\underline{\underline{\Psi}}_{R',R'}\frac{1}{2}]=2\underline{\underline{\Psi}}_{R,R'}$ applies in order to conserve agent number. The second term of this attendance equation [Eq. (2)] introduces a stochastic element in the game; it corresponds to the situation where agents may have several top scoring strategies and must thereby toss a coin to decide which to use. We note that Eq. (2) could be rewritten replacing the sgn function with tanh. The effect of this would be to make the number of agents playing strategy R_1 (as opposed to their other strategy R_2) vary smoothly as a function of the separation in the score of the two strategies, rather than simply playing the best. This modification is similar in concept to that of the thermal minority game (TMG) [8,9] wherein agents play their best strategy with a certain probability depending on its score. The difference here would be that, in contrast to the TMG, the system would still be entirely deterministic, hence lending itself readily to similar analysis as presented here.

With this formalism, the game can be described concisely by the following coupled mapping equations:

TABLE I. Percentage of time steps in which the minority room is changed by the stochastic decision of agents with tied strategies. Percentages are shown for the digital and proportional payoffs. Statistics obtained from 16 numerical runs of the MG with $N=101$, $s=2$, and over 1000 time steps.

m	$\chi=\mathrm{sgn}$	$\chi=1$
2	7.2 ± 4.2	0.7 ± 0.6
3	6.3 ± 3.0	2.4 ± 0.8
4	9.4 ± 2.1	3.4 ± 0.8

$$\underline{S}(t)=\underline{S}(t-1)-\underline{a}^{\mu(t-1)}\chi(A(\underline{S}(t-1),\mu(t-1))), \tag{3}$$

$$\mu(t)=2\mu(t-1)-PH\left(\mu(t-1)-\frac{P}{2}\right)+H(-A(\underline{S}(t-1),\mu(t-1))), \tag{4}$$

where $H(x)$ is the Heaviside function and $\chi(A)$ is a monotonic, increasing function of the game attendance quantifying the particular choice of reward structure (i.e., payoff). In most of the MG literature $\chi(A)=\mathrm{sgn}(A)$ or $\chi(A)=A$ [7]. Although the macroscopic statistical properties of the MG are largely unaltered by the choice of χ, we later demonstrate that the microscopic dynamics can be affected markedly.

This formulation shows that the MG obeys a one-step, stochastically disturbed deterministic mapping between states $\{\underline{S}(t),\mu(t)\}$ and $\{\underline{S}(t+1),\mu(t+1)\}$. It is interesting to ask the following question: "How important is the stochastic term of Eq. (2) to the resultant dynamics?" Table I shows the frequency with which the outcome [sgn($-A$)] is changed by the stochastic disturbance to the mapping. We can see that the stochastic term has a small but nonnegligible effect on the game. For the strategy reward system $\chi=\mathrm{sgn}$, the number of instances of coin tossing agents affecting the outcome is greater than with the proportional reward system of $\chi=1$. This is easily understood in terms of the homogeneity of the score vector $\underline{S}$; the $\chi=\mathrm{sgn}$ scoring system is much more likely to generate tied strategies than the $\chi=1$ system, which also incorporates the dynamics of the attendance A. Therefore, in the $\chi=\mathrm{sgn}$ scoring system there will be a much higher proportion of coin tossing agents and thus a greater effect on the game. We pause here to note that when the MG is modified to include a decaying "score memory," i.e., when Eq. (3) is modified to

$$\underline{S}(t)=\beta\underline{S}(t-1)-\underline{a}^{\mu(t-1)}\chi(A(\underline{S}(t-1),\mu(t-1))),$$

where $0<\beta<1$, then the chance of strategy scores being equal rapidly tends to zero with time and hence the game automatically can become completely deterministic.

The general effect of the stochastic contribution to the MG is to break the pattern of behavior emergent from the deterministic part of the map. It is therefore of great interest to examine further what the dynamics of this deterministic behavior are. To do this we replace the stochastic term of Eq. (2) by its mean. The equation thus becomes [$A_D(\underline{S},\mu)$ in Ref. [13]],

$$A(\underline{S},\mu)=\sum_{R=1}^{2P} a_R^\mu \sum_{R'=1}^{2P} [1+\mathrm{sgn}(S_R-S_{R'})]\underline{\underline{\Psi}}_{R,R'}. \quad (5)$$

Physically this replacement is an averaging process; when $S_{R_1}=S_{R_2}$ we have half the agents who hold $\{R_1,R_2\}$ playing R_1 and the other half playing R_2 [17]. Equations (3), (4), and (5) now define a deterministic map, which replicates the behavior of the MG between disturbances caused by the coin tossing agents; we refer to this system as the "deterministic minority game" (DMG). We will now use this system to investigate the emergence of microscopic and macroscopic dynamics.

III. DISORDER IN $\underline{\underline{\Psi}}$

The game is conditioned at the start with the initial state $\{\underline{S}(0),\mu(0)\}$. It is also given a $\underline{\underline{\Psi}}$ tensor for a particular parameter set N,m,s. The game's future behavior will be inherited from $\underline{\underline{\Psi}}$; games with sparsely and densely filled tensors hence behave in entirely different ways. By assuming that each entry of $\underline{\underline{\Omega}}$ is an independent binomial sample $\underline{\underline{\Omega}}_{R_1,R_2}=\mathrm{bin}[N,1/(2P)^2]$, we may categorize the disorder in the $\underline{\underline{\Omega}}$ tensor by the standard deviation of an element divided by its mean size. For $s=2$, this gives

$$\frac{\sigma(\underline{\underline{\Omega}}_{R_1,R_2})}{\mu(\underline{\underline{\Omega}}_{R_1,R_2})}=\sqrt{\frac{(2P)^2-1}{N}},$$

which rapidly becomes large as m increases. For low m and high N, the game is said to be in an "efficient phase" [2] where all states of the global-information set μ are visited equally and hence, on average, there is no drift in the strategies' scores, i.e., $\langle S_R\rangle_t=0$. In this regime, the disorder in the $\underline{\underline{\Omega}}$ tensor is small and thus all elements are approximately of equal magnitude. This in turn implies that the dynamics of the game are dominated by the movement of $\underline{S}$ rather than by the asymmetry of $\underline{\underline{\Omega}}$. The attendance of the $(s=2)$ game here reduces to

$$A(\underline{S},\mu)\approx\frac{N}{4P^2}\sum_{R=1}^{2P} a_R^\mu \sum_{R'=1}^{2P} \mathrm{sgn}(S_R-S_{R'}). \quad (6)$$

The second sum in Eq. (6) corresponds to a quantity q_R, which is based on the rank of strategy R; specifically $q_R=2P+1-2\rho_R$, where ρ_R is the rank of strategy R, with $\rho_R=1$ being the highest scoring and $\rho_R=2P$ being the lowest scoring. Hence Eq. (6) becomes

$$A(\underline{S},\mu)\approx\frac{N}{4P^2}\underline{a}^\mu\cdot\underline{q}. \quad (7)$$

We now examine the increment in strategy score, $\underline{\delta S}(t)=\underline{S}(t)-\underline{S}(t-1)$. For simplicity, we here assume the proportional scoring system of $\chi=1$. Hence

$$\underline{\delta S}=-\underline{a}^\mu A(\underline{S},\mu)\approx-\frac{N}{4P^2}\underline{a}^\mu(\underline{a}^\mu\cdot\underline{q}).$$

If we average over uniformly occurring states of μ, we then have for each strategy

$$\langle \delta S_R\rangle_\mu\approx-\frac{N}{4P^2}\sum_{R'=1}^{2P}\langle a_R^\mu a_{R'}^\mu\rangle_\mu q_{R'}.$$

We now use the orthogonality of strategies in the RSS,

$$\frac{1}{P}\sum_\mu a_{R_1}^\mu a_{R_2}^\mu\begin{cases}0 & \text{for } R_1\neq R_2,\\ 1 & \text{for } R_1=R_2,\\ -1 & \text{for } R_2=\bar{R}_1.\end{cases}$$

This yields

$$\langle \delta S_R\rangle_\mu\approx\frac{N}{2P^2}\{\rho_R-\rho_{\bar{R}}\}, \quad (8)$$

where $\bar{R}=2P+1-R$ is the anticorrelated strategy to R. Equation (8) now shows us explicitly that strategies and their anticorrelated partners attract each other in pairs. The magnitude of the score increment is also of interest; for low m and high N the attractive force is large, which will cause the strategies to overshoot each other and thus perform a constant cycle of swapping positions. As we increase m or decrease N, the attractive force becomes weaker and so the score cycling adopts a longer time period; it eventually becomes too weak to overcome the separate force arising from the asymmetry in $\underline{\underline{\Psi}}$. Hence the system moves away from the strongly mean reverting behavior in $\underline{S}$.

We can investigate this change of regime further by examining $\langle \delta S_R\rangle_\mu$ for finite disorder in $\underline{\underline{\Psi}}$ (see note on validity of averaging [18]). Again using the orthogonality of strategies in the RSS, we have

$$\langle \delta S_R\rangle_\mu=\delta S_R^{\mathrm{bias}}+\delta S_R^{\mathrm{restoring}}$$

$$=-\sum_{R'=1}^{2P}(\underline{\underline{\Psi}}_{R,R'}-\underline{\underline{\Psi}}_{\bar{R},R'}), \quad (9)$$

$$-\sum_{R'=1}^{2P}[\mathrm{sgn}(S_R-S_{R'})\underline{\underline{\Psi}}_{R,R'} + \mathrm{sgn}(S_R+S_{R'})\underline{\underline{\Psi}}_{R,R'}]. \quad (10)$$

Equation (9) has two distinct contributions. The first term $\delta S_R^{\mathrm{bias}}$ arises from disorder in $\underline{\underline{\Psi}}$ alone and is time independent, representing a constant bias on the score increment. The second term $\delta S_R^{\mathrm{restoring}}$ acts as a mean reverting force on the strategy score; its magnitude depends on how many strategies lie between it and its anticorrelated partner [just as in Eq. (8)]. Figure 1 illustrates this for a case where $S_R>0$; here the net contribution to $\delta S_R^{\mathrm{restoring}}$ is likely to be negative as there are more contributing elements with a negative sign than with a positive sign. The strategies $R'\ni-|S_R|<S_{R'}<|S_R|$ always contribute terms $-\mathrm{sgn}(S_R)(\Psi_{R,R'}+\Psi_{\bar{R},R'})$ to $\delta S_R^{\mathrm{restoring}}$ and so will always act as a mean reverting component. Terms from strategies outside this range will always be divided into equally sized positive and negative groups as

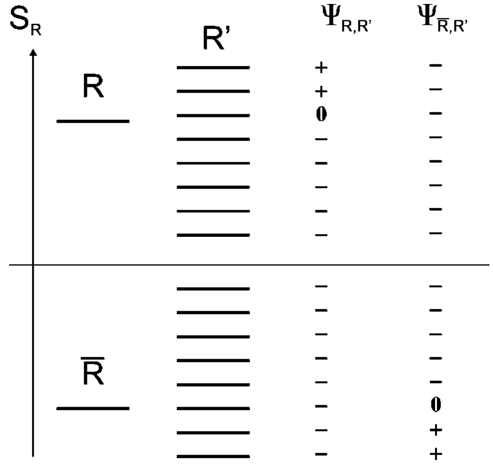

FIG. 1. Schematic representation of the signs of contributing terms to $\delta S^{\mathrm{restoring}}$.

shown in Fig. 1. These groups will on average cancel out each other's effect on the score increment.

We can model the average magnitude of each term in Eq. (9) by using the same binomial representation for the elements of $\underline{\underline{\Omega}}$ as before. The mean magnitude of the bias and restoring-force terms $\langle|\delta S_R^{\mathrm{bias}}|\rangle_R$ and $\langle\langle|\delta S_R^{\mathrm{restoring}}|\rangle_{S_R}\rangle_R$ are thus approximately given as follows:

$$\langle|S_R^{\mathrm{bias}}|\rangle_R \approx \sqrt{(N/P\pi)[1-1/(2P)^2]},$$

$$\langle\langle|\delta S_R^{\mathrm{restoring}}|\rangle_{S_R}\rangle_R \approx \frac{N\gamma}{4P^2}. \tag{11}$$

The term γ enumerates the average net number of terms in $\delta S_R^{\mathrm{restoring}}$ that act to mean revert S_R, i.e., the excess number of terms with negative sign, $-\mathrm{sgn}(S_R)$. Averaged over the entire set of strategies, we have $\gamma = 2P$. Figure 2 shows that our approximate form for the average strategy score bias in Eq. (11) is extremely good over the entire range of $\alpha = P/N$ whereas the approximation of the restoring-force term becomes progressively worse as α is increased. This effect can be explained in terms of the "market impact" of a strategy. The greater the number of agents using a strategy

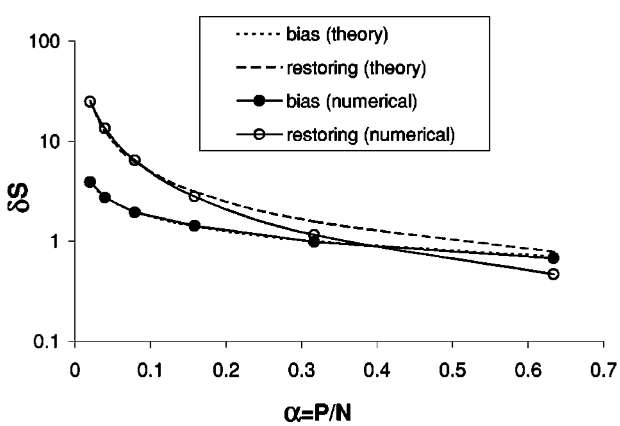

FIG. 2. Numerical and approximate analytical magnitude of average score increment terms $\langle|\delta S_R^{\mathrm{bias}}|\rangle_R$ and $\langle\langle|\delta S_R^{\mathrm{restoring}}|\rangle_{S_R}\rangle_R$

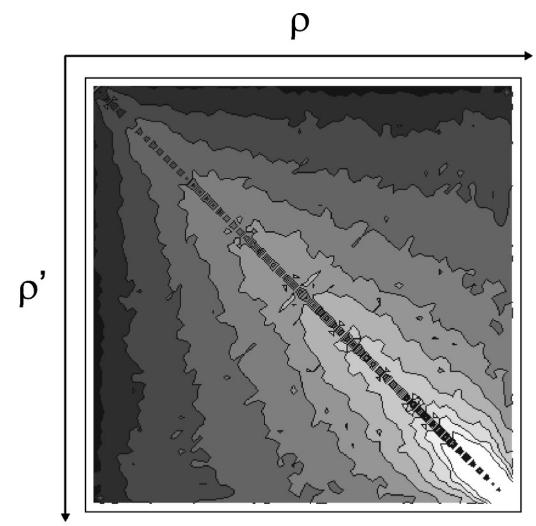

FIG. 3. Contour plot of $\langle\Psi_{\rho,\rho'}\rangle$, i.e., an average of the strategy population tensor reordered each turn with strategies running from highest to lowest score (top to bottom and left to right). Black areas indicate low population and white areas indicate high population. The averaging is carried out over 50 runs (different $\underline{\underline{\Omega}}$) and 1000 turns within each run. MG game parameters $\alpha = 0.32$, $s = 2$.

n_R, the greater its contribution is to the attendance as can be seen from Eq. (1). As n_R is increased above $n_{R' \neq R}$, the probability of the game outcome $[-\mathrm{sgn}(A)]$ being opposed to a_R^{μ} becomes greater and hence strategy R being penalized is also more probable. This effect will arise if the quenched disorder in $\underline{\underline{\Psi}}$ is such that more agents hold strategy R than $R' \neq R$. As α is raised and the quenched disorder in $\underline{\underline{\Psi}}$ grows, this effect will become increasingly important. Hence it can be seen that $\underline{\underline{\Psi}}_{R,R'}$ and $\{S_R, S_{R'}\}$ are not independent as assumed in obtaining Eq. (11), but are instead correlated through the effect of market impact; this correlation becomes more significant as α is increased.

The nature of the correlation between $\underline{\underline{\Psi}}_{R,R'}$ and $\{S_R, S_{R'}\}$ introduced by market impact is nontrivial in form as can be seen from Fig. 3. We will not discuss the details of an analytic reconstruction of $\underline{\underline{\Psi}}_{\rho,\rho'}$ here, but will instead simply note some straightforward constraints on its form. Let us take the approximation that on average, the ranking of the strategies $\{\rho_R\}$ is given by the ranking of their bias terms $\{\delta S_R^{\mathrm{bias}}\}$. This will be true *on average* for a system described by Eq. (9). We then use the approximation that $\delta S_R^{\mathrm{bias}} \sim N[0, \sqrt{(N/2P)(1-1/4P^2)}]$. Ordering the bias terms, resulting from samples drawn from this distribution, gives us that $\underline{\underline{\Psi}}_{\rho,\rho'}$ satisfies

$$\mathrm{Erf}\left(\frac{\langle\delta S_\rho^{\mathrm{bias}}\rangle}{\sqrt{[N/P][1-1/4P^2]}}\right) = \frac{P-\rho}{P},$$

with $\delta S_\rho^{\mathrm{bias}}$ given by $-\Sigma_{\rho'=1}^{2P}(\underline{\underline{\Psi}}_{\rho,\rho'} - \underline{\underline{\Psi}}_{\bar{\rho},\rho'})$, as in Eq. (9). This relation gives us an indication of how the rank of a strategy is affected by its excess population, and is consistent with the form of $\underline{\underline{\Psi}}_{\rho,\rho'}$ as shown in Fig. 3. Note that in the absence of market impact we would not be able to write down any equation linking these parameters and Fig. 3 would be flat with no structure.

We have thus shown how market impact is profoundly manifest within the structure of the MG [1]. In particular, Fig. 2 shows clearly that consideration of market impact is necessary in the calculation of the transition point from efficient to inefficient regimes [1]. The game enters the inefficient regime if the magnitude of the bias term to the score increment (arising from disorder in $\underline{\underline{\Psi}}$) exceeds the magnitude of the restoring-force term. We can calculate when on average strategies begin to drift by looking at when $\langle|\delta S_R^{\mathrm{bias}}|\rangle_R = \langle\langle|\delta S_R^{\mathrm{restoring}}|\rangle_{S_R}\rangle_R$ in Eq. (11). This occurs near $\alpha=\alpha_c\approx\pi/4$. This overestimation of the transition point (which numerically occurs in the DMG at around $\alpha_c=0.39$) could be corrected by taking into account the nonflat structure of $\underline{\underline{\Psi}}_{\rho,\rho'}$. We would like to stress here that only on average does there exist a specific point at which the game passes from mean reverting to biased behavior (efficient to inefficient regime). Because the behavior of the game is dictated by the disorder in $\underline{\underline{\Omega}}$ and not just by the specific parameters N,m,s alone, a knowledge of α is not enough information to classify the game as being in either the efficient or inefficient regime. The value of α_c cannot therefore be considered a ''critical'' value in this system away from the thermodynamic limit of large N and P.

Equation (9) can also yield insight into the dynamics in the regime past the transition point. We were able to predict from Eq. (8) that in the efficient regime, pairs of anticorrelated strategies would cycle around each other thus producing an everchanging score rank vector $\underline{\rho}$. In the inefficient regime wherein the strategy scores have appreciable bias, it would be natural to assume that $\underline{\rho}$ would rapidly find a steady state as the strategy scores diverged. This in fact does not happen; for example, consider the outermost pair of strategies in the score space (i.e., the current best and its anticorrelated partner, the worst) at a point in the game. For these strategies, Eq. (9) is given approximately by

$$\langle\delta S_R\rangle_\mu \approx -\sum_{R'=1}^{2P}(\underline{\underline{\Psi}}_{R,R'}-\underline{\underline{\Psi}}_{\bar{R},R'}) - \mathrm{sgn}(S_R)\sum_{R'=1}^{2P}(\underline{\underline{\Psi}}_{R,R'}+\underline{\underline{\Psi}}_{\bar{R},R'}).$$

Irrespective of the disorder in $\underline{\underline{\Psi}}$, we have $|\delta S^{\mathrm{bias}}| \lesssim |\delta S^{\mathrm{restoring}}|$. It is thus likely that this strategy pair will attract each other until at least one other pair takes their place as best/worst. This behavior will lead to a nonstationary $\underline{\rho}$ vector even in this regime.

The present analysis has described general properties of the game such as the transition in behavior between efficient and inefficient regimes. It has also shown that dynamical processes such as the changing nature of $\underline{\rho}$ can be quantitatively explained purely in terms of the quenched disorder in the strategy population tensor $\underline{\underline{\Omega}}$.

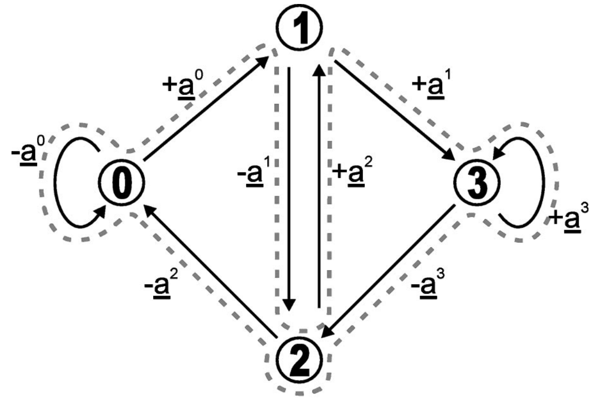

FIG. 4. De Bruijn graph D_2 [2] corresponding to $m=2$. Vertices are labeled with the state μ, edges are labeled with the quantity $\overline{\delta S}/|\chi(A)|$. The dotted line shows one of the two possible Eulerian trails of this graph.

IV. DYNAMICS IN μ SPACE

The previous section was concerned with the behavior of the strategy-score vector $\underline{S}$, and often treated the dynamical variable μ as a random process to be averaged over. This however glosses over the subtle and very interesting dynamics of μ itself as dictated by Eq. (4). (References [10,13] also consider aspects of μ dynamics.) To aid in our discussion, we note that Eq. (4) describes a trajectory along the edges of a directed de Bruijn graph $D_2(m)$. Figure 4 shows an example of such a graph for $m=2$. As explained in the previous section, in the efficient regime $\underline{S}$ is strongly mean reverting. This implies that the set of states of the game $\{\underline{S},\mu\}$ is finite. As the system is Markovian and deterministic, this in turn implies that it must exhibit periodic behavior in this regime as return to a past state would then be followed by the revisiting of the trajectory from that state. In the inefficient regime where the strategy scores are biased, the set of states $\{\underline{S},\mu\}$ is unbounded and we may expect aperiodic behavior of the DMG.

We now examine the structure of the periodic behavior in the efficient regime. One observation from numerical simulations is that the period, i.e., return time to any state $\{\underline{S},\mu\}$ is observed over many runs to be $T=2P$ for the $\chi=\mathrm{sgn}$ scoring system whereas for the $\chi=1$ system the period is much longer and run dependent. This periodic behavior seems able to exist up to the point where the occurrence of zero attendance $A(\underline{S},\mu)=0$ causes stochastic disturbance to μ [17]; after this point we can no longer treat our system as deterministic. Such periodic behavior must satisfy the conditions $\{\underline{\Delta S}_{\mathrm{cycle}}=0,\ \Delta\mu_{\mathrm{cycle}}=0\}$. $T=2P$ is in fact the shortest possible period that satisfies these conditions. The two edges leading away from any vertex μ on the de Bruijn graph must necessarily incur score increments of the opposite sign: $+\underline{a}^{\mu}|\chi(A)|, -\underline{a}^{\mu}|\chi(A)|$ corresponding to positive and negative attendance, respectively. The vectors $\underline{a}^{\mu 1}$ and $\underline{a}^{\mu 2\neq\mu 1}$ are orthogonal; hence the only way that an increment to the score of $\underline{a}^{\mu[t]}\chi(A(\underline{S}(t),\mu(t)))$ can be negated in order to achieve $\underline{\Delta S}_{\mathrm{cycle}}=0$, is to return to that vertex (i.e.), $\mu(t')=\mu(t)$ a particular number of times such that

$$\sum_{(t')} \chi(A(\underset{\sim}{S}(t'),\mu(t)))=0. \tag{12}$$

This condition must be satisfied at all vertices of the graph because the set $\{t'\}$, which satisfies Eq. (12) must have a minimum of two entries (each of opposite attendance) thereby leading the game to different, new vertices until all are spanned.

Consider the $\chi=\text{sgn}$ scoring system. The condition corresponding to Eq. (12) is easily satisfied at each vertex with a set $\{t'\}$ of exactly 2λ entries, λ being an integer. We now have the situation where all edges of the graph are visited equally. The shortest way of doing this is with $\lambda=1$; this cycle is known as an "Eulerian trail." This dynamical stable state of the game acts as an attractor; the MG in the efficient phase will rapidly find this state after undergoing a stochastic disturbance. We note that the time-horizon minority game [13] exhibits similar behavior for special values of the time horizon τ. This trajectory of the DMG along a Eulerian trail corresponds to the occurrence of perfect antipersistence in the $(A|\mu)$ time series. This antipersistence has been empirically observed in many studies of the MG [1,7,15].

Now consider the $\chi=1$ scoring system. The condition of Eq. (12) is very much harder to achieve over all vertices as the dynamics of A are incorporated back into the score vector $\underset{\sim}{S}$ making the set $\{\underset{\sim}{S},\mu\}$ very much larger. This explains the very much longer period of this game, which, even over very long time windows, can appear aperiodic. The Eulerian trail will still, however, be an attractor to the dynamics within μ space, since the antipersistence in $(A|\mu)$ is still strong (in the efficient phase). It is not however perfect as was the case for the DMG using the $\chi=\text{sgn}$ scoring system.

To quantitatively explain this antipersistence, we make the following approximation:

$$\text{sgn}(A)\approx\text{sgn}(\underset{\sim}{a}\cdot\underset{\sim}{S}). \tag{13}$$

This approximation can be understood by referring back to Eq. (7) where $\underset{\sim}{S}$ now plays the same role as the rank measure $\underset{\sim}{q}$. It is valid for the regime where the strategy scores are densely spaced, i.e., for the efficient regime/low disorder in $\underset{\approx}{\Psi}$. Consider the $\chi=\text{sgn}$ scoring system wherein the score vector is simply given by $\underset{\sim}{S}(t)=\underset{\sim}{S}(0)-\Sigma_{j=1}^{t-1}\text{sgn}\,(A(j))\underset{\sim}{a}^{\mu(j)}$. We use the fact that the vectors $\underset{\sim}{a}^{\mu1}$ and $\underset{\sim}{a}^{\mu2\neq\mu1}$ are orthogonal to transform Eq. (13) to the following form:

$$\text{sgn}(A(t))\approx\text{sgn}\left(\underset{\sim}{a}^{\mu}\cdot\underset{\sim}{S}(0)-2P\sum_{\{t'\}}\text{sgn}(A(t))\right), \tag{14}$$

where we recall that the set of times $\{t'\}$ are such that $\mu(t')=\mu(t)=\mu$ for $0<t'<t$. This dynamical process occurring over times t' rather than t is zero reverting. Let us demonstrate this by taking an example. Let $P=4$ and the initial strategy score be such that $\underset{\sim}{a}^{\mu}\underset{\sim}{S}(0)=20$. The time series of $\text{sgn}\,(A(t))$ thus becomes as shown in Table II. Hence the game cascades from its initial state, the attendance at a given vertex of the de Bruijn graph ($[A|\mu]$) exhibiting persistent behavior until a point is reached such that $|\underset{\sim}{a}^{\mu}\cdot\underset{\sim}{S}(0)-2P\Sigma_{\{t'\}}\text{sgn}\,(A(t))|<2P$. Subsequently the attendance $(A|\mu)$ becomes perfectly *antipersistent*. When this antipersistence occurs at each vertex, the game has locked into one of the $2^{2^m}/2^{m+1}$ Eulerian trails. The analysis above can be generalized for different scoring systems (such as $\chi=1$) where, in general, it is found that the game exhibits strong but not perfect antipersistence in $(A|\mu)$ in this regime.

TABLE II. An example of how the game cascades from an initial state [cf. Eq. (14)]. Here $P=4$ and $\underset{\sim}{a}^{\mu}\cdot\underset{\sim}{S}(0)=20$. The attendance (right column) exhibits persistent, and then antipersistent, behavior.

$\underset{\sim}{a}^{\mu}\cdot\underset{\sim}{S}(0)-2P\Sigma_{\{t'\}}\text{sgn}(A(t'))$	$\text{sgn}(A(t))$
$20-8\times(0)=20$	1
$20-8\times(0+1)=12$	1
$20-8\times(0+1+1)=4$	1
$20-8\times(0+1+1+1)=-4$	-1
$20-8\times(0+1+1+1-1)=4$	1
$20-8\times(0+1+1+1-1+1)=-4$	-1

In the analysis above we introduced the effect of the initial condition on the score vector $\underset{\sim}{S}(0)$ (see also Ref. [14]). However, we could just as correctly view $\underset{\sim}{S}(0)$ as the current state, left by some other game process such as a shock to the system, a buildup from some other game mechanism or a stochastic perturbation. It is therefore interesting to examine how the DMG evolves after a given state $\{\underset{\sim}{S}(0),\mu(0)\}$ is imposed. The "initial" condition $\underset{\sim}{S}(0)$ must obey the form $S_R=-S_{\bar{R}}$; this is to ensure that *a priori* no strategies are given a bias. It would be unphysical to break this rule; strategy R always loses the same number of points as its anticorrelated partner $\bar{R}$ gains in any reasonable physical mechanism. We expect that if the elements $S_R(0)$ have magnitude less than $2P$, then the system will very quickly lock into the Eulerian trail trajectory and visit all μ states equally. However, if the elements $|S_R(0)|\geqslant 2P$ then Eq. (14) predicts that there will be persistence in $(A|\mu)$ until the dynamical stable state is found. This persistence in trajectory at each node of the de Bruijn graph will lead to the game visiting only a small subset of the vertices on the graph unlike in the stable state situation. This reduced cycling effect may lead to a bias in the attendance over a significant period of time, i.e., a "crash" or "rally."

We now demonstrate the recovery of the DMG from a randomly chosen initial score vector $\underset{\sim}{S}(0)$. We take a system with low disorder in $\underset{\approx}{\Psi}$ and $m=2$ (such that $2P=8$). However, we draw $S_R(0)$ from a much wider uniform distribution spanning -100 to 100. (Note we maintain $S_R=-S_{\bar{R}}$ as required.) Figure 5 shows the evolution of the game out of this state. The initial condition is soon "worked out" of the system—it rapidly finds the Eulerian cycle $\mu=0,0,1,3,3,2,1,2,\ldots$, after only 174 turns. As can also be seen, the game adopts several different types of cycles on its way towards this stable state. The switch between cycle types occurs as each vertex snaps from persistent to antipersistent behavior.

We have hence discussed and explained the dynamics of

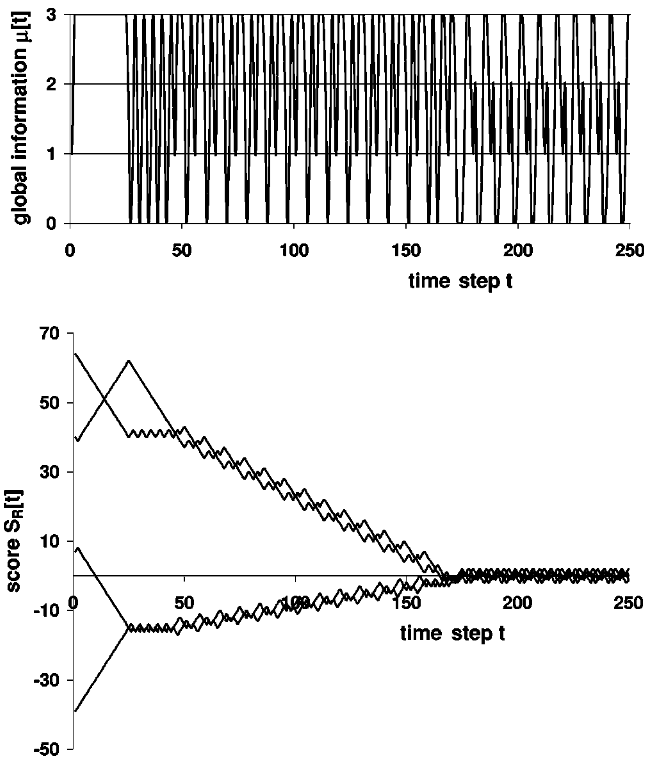

FIG. 5. An example of the convergence of the DMG onto the Eulerian trail attractor. Top graph shows the dynamics in the global information $\mu(t)$. Bottom graph shows the dynamics in score $S_R(t)$ for $1 \leqslant R \leqslant 4$ (out of $2P=8$). Game locks into attractor at turn 174. Game parameters $N=101$, $m=2$, $s=2$.

the stable state and how the system enters that state from an initial or perturbed state. This analysis has been done for the system in the efficient phase where the quenched disorder of $\underline{\underline{\Psi}}$ is low. The inefficient regime will, in general, show a different set of dynamics. As discussed earlier, the inefficient phase is characterized by score vectors that have an appreciable drift; this is an effect of the disorder in $\underline{\underline{\Psi}}$. The corresponding unbounded $\underline{S}(t)$ vector leads to an unbounded set of states for the system $\{\underline{S},\mu\}$. This suggests that the overall dynamics may be aperiodic, i.e., the system never returns to a past state. We can however say something about the nature of the resulting dynamics in μ space. As the score vector diverges the score rank vector $\underline{\rho}$ becomes more well defined (although not completely stationary in time, as mentioned in Sec. III). This is tantamount to there being a certain degree of persistence in the attendance at a vertex $(A|\mu)$. This will lead to the motion around the de Bruijn graph being limited to a certain subspace, just as that described above for the recovery from an initial score vector $\underline{S}(0)$. This difference in the dynamics for the efficient and inefficient regimes leads to the well-documented result that the occurrence of different $(m+1)$ bit words is even in the efficient regime but very uneven in the inefficient regime [7].

V. CONCLUSION

The results in this paper confirm that the MG can be usefully viewed as a stochastically disturbed deterministic system, and that this deterministic system can be described concisely by coupled mapping equations [Eqs. (3), (4), and (5)]. We used this system to explore the dynamics of the score vector $\underline{S}(t)$. We showed that the score increment comprises a bias and restoring-force term, the comparative magnitude of these terms being governed by the disorder in the strategy population tensor $\underline{\underline{\Omega}}$. Furthermore, we were able to obtain analytic approximations for the bias and restoring-force terms. We showed how the market impact effect correlated the strategy population to the score vector and how this then affected our approximations.

We also discussed the dynamics of the global information $\mu(t)$ as a trajectory on a de Bruijn graph. We were able to show that in the efficient regime the system would be periodic and that the favored periodic trajectory was that of an Eulerian trail. Analytically we were able to demonstrate how antipersistence and persistence arise in the attendance at a vertex $(A|\mu)$, and how this would manifest itself in efficient and inefficient regimes either in response to a perturbed state or an initial condition of $\underline{S}(0)$.

In short, this analytic treatment has not only again explained why MG systems cross over from efficient to inefficient behavior (this effect was explained here simply in terms of the quenched disorder and in absence of the thermodynamic limit), we have also shown how it is possible to unravel the rich dynamics and explore effects that happen within a given realization of the system and not simply on average. The analytical treatment of this work may also be easily used to investigate extensions to the model. For instance, the "grand-canonical MG" [5], used as a model for the financial markets, represents a minimal alteration to Eq. (5). By using the same techniques as presented here we can investigate effects in this modified model such as how large drawdowns occur and how stability and volatility are effected by both parameter choice and external perturbation.

ACKNOWLEDGMENTS

We are grateful to A. Short and P. M. Hui for useful discussions and comments.

[1] D. Challet and Y. C. Zhang, Physica A **246**, 407 (1997); **269**, 30 (1999); D. Challet and M. Marsili, Phys. Rev. E **60**, R6271 (1999); D. Challet, M. Marsili, and R. Zecchina, Phys. Rev. Lett. **84**, 1824 (2000); M. Marsili, D. Challet, and R. Zecchina, e-print cond-mat/9908480.

[2] See http://www.unifr.ch/econophysics for a detailed account of the minority game literature.

[3] D. Challet and Y. C. Zhang, Physica A **256**, 514 (1998).

[4] M. Hart, P. Jefferies, N. F. Johnson, and P. M. Hui, e-print cond-mat/0008385.

[5] P. Jefferies, M. L. Hart, P. M. Hui, and N. F. Johnson, Eur. Phys. J. B **20**, 493 (2001).

[6] M. Hart, P. Jefferies, N. F. Johnson, and P. M. Hui, e-print cond-mat/0005152; e-print cond-mat/0003486; N. F. Johnson, P. M. Hui, D. Zheng, and M. Hart, J. Phys. A **32**, L427 (1999); N. F. Johnson, M. Hart, and P. M. Hui, Physica A **269**, 1 (1999).

[7] R. Savit, R. Manuca, and R. Riolo, Phys. Rev. Lett. **82**, 2203 (1999); Physica A **276**, 234 (2000); **276**, 265 (2000). See also Y. Li, A. VanDeemen, and R. Savit, e-print nlin.AO/0002004. This group's work shows that similar statistical properties arise for different payoff functions. This can be explained using the crowd-anticrowd theory of Refs. [4,5], since the probability function $P(r'=\overline{r})$ has a similar form for a wide range of payoff functions [P. M. Hui *et al.* (unpublished)].

[8] A. Cavagna, J. P. Garrahan, I. Giardina, and D. Sherrington, Phys. Rev. Lett. **83**, 4429 (1999); J. P. Garrahan, E. Moro, and D. Sherrington, e-print cond-mat/0012269.

[9] M. Hart, P. Jefferies, N. F. Johnson, and P. M. Hui, Phys. Rev. E **63**, 017102 (2001); P. Jefferies, M. Hart, N. F. Johnson, and P. M. Hui, J. Phys. A **33**, L409 (2000).

[10] M. Marsili and D. Challet, e-print cond-mat/0004196.

[11] A. Cavagna, Phys. Rev. E **59**, R3783 (1999).

[12] J. A. F. Heimel and A. C. C. Coolen, e-print cond-mat/0012045.

[13] M. L. Hart, P. Jefferies, and N. F. Johnson, e-print cond-mat/0102384.

[14] M. Marsili and D. Challet, e-print cond-mat/0102257.

[15] D. Zheng and B. H. Wang, e-print cond-mat/0101225.

[16] The expression $\underline{\underline{\Omega}}+\underline{\underline{\Omega}}^T$ double counts the diagonal terms of $\underline{\underline{\Omega}}$. This property is compensated for by the form of Eq. (2).

[17] Partitioning the number of agents with tied strategies in this way may lead to noninteger n_R. In turn a zero attendance is possible even from an odd number of agents. In these (rare) occurrences we assign $A=\pm 1$ randomly. This break in the determinism of the system is of particular relevance to the discussion in Sec. IV.

[18] As the game is moved away from the efficient regime wherein states of μ are visited equally, averaging over μ should strictly become invalid. However, some average properties of the game are reasonably insensitive to the replacement of the μ process of Eq. (4) by a uniform random process. See Refs. [10,11] for a discussion of the validity of random μ.